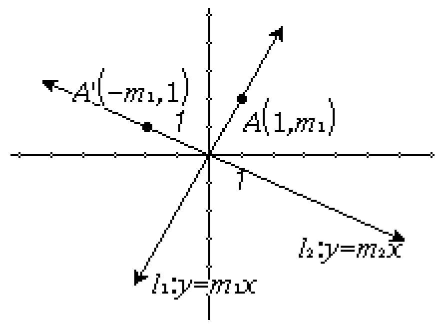
8-4: The Slopes of Perpendicular Lines

Let l_1 be a line whose equation is $y = m_1 x$, $m_1 \neq 0$. Then O(0, 0) and A(1, m_1) are two points on the line. The slope of l_1 = m_1.



- Under a counterclockwise rotation of 90° about the origin, the image of A is $A'(-m_1, 1)$. Since $\angle AOA'$ is a right angle, ______________________________ .
- Let l_2 be the line through the origin and the point A'(-m_1, 1), and let the slope of l_2 = m_2. Then:

$$m_2 = \frac{0-1}{0-(-m_1)}$$
$$= \frac{-1}{m_1}$$

We have shown that when two lines through the origin are perpendicular, the slope of one is the negative reciprocal of the slope of the other. Is the rule true for the slopes of any two perpendicular lines? To show this we need to prove the following theorem.

Theorem: Under a translation, slope is preserved, that is, if line *l* has slope *m*, then under a translation, the image of *l* also has slope *m*.

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on line *l*. Then:

slope of *l* =

Under a translation $T_{(a,b)}$ the images of *P* and *Q* are $P'(x_1 + a, y_1 + b)$ and $Q'(x_2 + a, y_2 + b)$. Therefore the slope of *l*', the image of *l* is:

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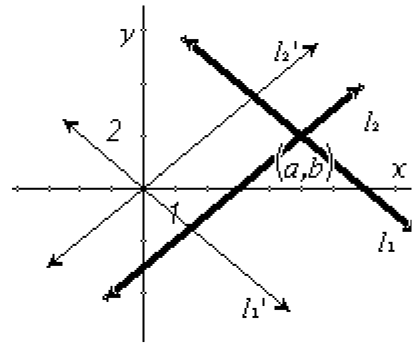
- As a simple application of this theorem, we can show that the slopes of any two perpendicular lines are negative reciprocals of each other.

Theorem: If two non-vertical lines are perpendicular, then the slope of one is the negative reciprocal of the other.

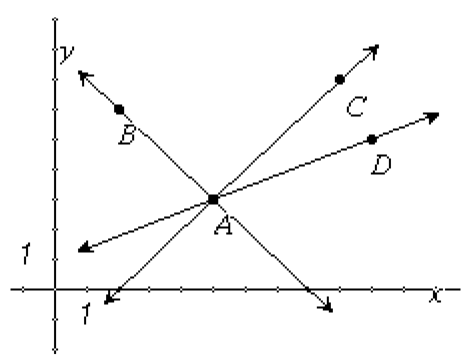


Proof: Let l_1 and **Error! Objects cannot be created from editing field codes.** be two lines that intersect at (*a*, *b*). Under the translation $(x, y) \rightarrow (x-a, y-b)$, the image of (*a*, *b*) is _______________ .

If the slope of is *m*, then by the above theorem, the slope of its image l_1' is ______. Since l_1 and **Error! Objects cannot be created from editing field codes.** are perpendicular, their images l_1' and **Error! Objects cannot be created from editing field codes.**' are also perpendicular because translations preserve ___________________________________. Using what we established at the beginning, if the slope of l_1' is *m*, the slope of **Error! Objects cannot be created from editing field codes.**' is ____________. Slope is preserved under translation. Therefore the slope of **Error! Objects cannot be created from editing field codes.** is ____________.

Is the converse true? We will use an **indirect proof**:

Theorem: If the slopes of two lines are negative reciprocals, then the two lines are perpendicular.



Given: Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ that intersect at *A*. The slope of $\overleftrightarrow{AB}$ is *m*, and the slope of $\overleftrightarrow{AC}$ is $-\frac{1}{m}$, the negative reciprocal of *m*.

Prove: $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$

Proof:

Statements	***Reasons***
1. $\overleftrightarrow{AB}$ is **not** perpendicular to $\overleftrightarrow{AC}$	**Assumption**
2. **Construct** $\overleftrightarrow{AD}$ perpendicular to $\overleftrightarrow{AB}$ at A	At a given point on a given line, one and only one perpendicular line can be drawn.
3. The slope of $\overleftrightarrow{AB}$ is m,	
4. The slope of $\overleftrightarrow{AD}$ is _________	
5.	Given
6. A, C, and D are on the same line, that is $\overleftrightarrow{AC}$ and $\overleftrightarrow{AD}$ are the same line.	Three points lie on the same line if and only if:
$\therefore$	**Contradiction (steps 1, 6)**

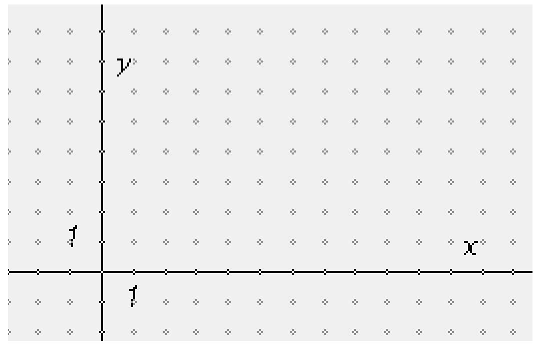
- As a biconditional:

Theorem: Two non-vertical lines are perpendicular if and only if the slope of one is the negative reciprocal of the other.

Example:
What is the slope of l_2, the line perpendicular to l_1 if the equation of l_1 is $x + 2y = 4$?

Example:

Show that ΔABC is a right triangle if its vertices are $A(1, 1)$, $B(4, 3)$, and $C(2, 6)$.



Homework: